**Lecture Note-7**

**Integration using Cauchy’s Residue** **Theorem (CRT)**

Two main reasons account for the importance of integration in the complex plane. The practical reason is that complex integration can evaluate certain real integrals appearing in applications that are not accessible by real integral calculus. The theoretical reason is that some basic properties of analytic functions are difficult to prove by other methods. Complex integration also plays an important role in connections with special function, such as the gamma function, the error function, various polynomials and others, and the application of these functions in physics.

**Cauchy’s Integral Formula:**

If a function  is analytic within and on a simple closed contour  and if  is any point interior to then,

**- - - - - - - - - (1)**

**Special case :** If is not an interior point of the contour  then **.**

Differentiating n-1 times w.r.to

****

**Definition of singular point (of an analytic function):**

A point at which an analytic function is not defined, i.e., at which fails to exist, called a singular point or pole or singularity of the function.

**Example 7.1:** If then , 3 are the singular points of 

**Residue Finding Method:**

If is analytic inside and on a simple closed curve *C* except at pole or has singularity at  of order 1, then

.

If is analytic inside and on a simple closed curve *C* except at pole or has singularity at of order ***m***, then



**Cauchy Residue Theorem:**

If is analytic inside and on a simple closed curve *C* except at a finite number of singular points  inside *C*, then

|  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| **Example 7.2:** Evaluate by CRT , where C is the circle  **Solution**: For singular point,    Singular point is a pole of order 2. The point lies inside the circle .  Residue at the point  is,          So, by CRT we know,    **Example 7.3:** Evaluate the contour integral by CRT, where C is the circle   |  |  | | --- | --- | | **Solution:** Thepoles or singularities of  are as follows:  A pole of order 3 at . This pole lies inside the contour C.  Residue at the point of order 3 is given by        So by CRT we know, =0. |  |   **Sample Exercise Set on Cauchy residue theorem: 7.1**     1. (i) Find all the singular points of the following functions , f(z) and show the points in the argand diagram, where f(z) =, .   (ii) Find all the singular points of the following functions , *f(z)* and show them in the argand diagram,then find corresponding residues : f(z) = , .   1. State Cauchy’s integral formula and Cauchy’s residue theorem (CRT). For each of the followings sketch the indicated path and hence evaluate applying Cauchy’s residue theorem (CRT), ( if possible):   (a) , is the circle .  (b) , consists of .  (c) , where C is the circle  3. Evaluate the followings applying Cauchy’s residue theorem (CRT) (if possible):  (a) Evaluate the integrals along the contour as given in the figures:  (i) (Fig. 1), (ii)(Fig. 2), (iii) (Fig. 3).  C:\Users\Miraz\AppData\Local\Microsoft\Windows\Temporary Internet Files\Content.Word\Untitled.jpg  Fig.1 Fig.2 Fig.3  4. For the followings sketch the indicated path and hence evaluate applying Cauchy’s residue theorem (CRT) if possible:  (a) , is the contour as (i) , (ii).  (b)where is the circle  (c) where is the circle .  **Application of Residue Theorem**  **Evaluation of Real Definite Integrals by Contour Integrals:**  A large number of real definite integrals, whose evaluation by usual methods become sometimes very tedious, can be easily evaluated by using **Cauchy’s Residue theorem**. For finding the integrals we take a suitable complex function and closed curve , then find the poles or singularity of the function and calculate residues at those poles only which lie within the curve . Then using Cauchy’s residue theorem we have    We call the curve, a contour and the process of integration along a contour is called contour integration.  **(Improper Integral)** Infinite real integrals of the form  **or,** whereand are polynomials in . Such integrals can be reduced to contour integrals, if   1. has no real roots. 2. The degree of is greater than that of by at least two.   **Procedure to solve:**   |  |  | | --- | --- | | To evaluate such integrals we consider the contour integrals  where C is the closed contour, consisting the real axis from to and the upper half of the circle i.e.,  - - - - (1) |  |   Now using CRT we get,    Then (1) becomes,    - -- -- --- --- (2)    Now, on the semi circular path , . Then applying Jordan’s Lemma,    Then (2) reduces to    **Example 7.5:** Evaluate by using contour integration.  **Solution:**    We consider where C is the closed contour consisting of the semi circle  of radius R together with the part of the real axis –R to +R. i.e.,  ……(1)  Now the first integral has singularities or pole at i.e. of order 2. But the only pole is inside the contour C. So,      So by CRT,    So equation (1) becomes    By Jordan Lemma letting and noting that the second integral in left hand side would become zero.    Hence,    **Example 7.6:** Evaluate by using contour integration.  **Solution:** We consider where *C* is the closed contour consisting of the semi circle  of radius *R* together with the part of the real axis *–R* to *+R*. i.e.,  ……(1)  The figure in the previous example should be considered here.  Now the first integral has singularities or poles at        When        i.e. there are four poles, but only two poles at and lie within the contour C. So,          So by CRT,    So equation (1) becomes  .  By Jordan Lemma letting and noting that the second integral in left hand side would become zero. Hence,    Matlab command for improper integral:   |  |  | | --- | --- | | 1. Evaluate ,   >> fun=@(x) 1./(x.^2+1);  >> q=integral(fun,0,inf)  q = 1.5708 | 1. Evaluate ,   >> f=@(x) 1./(x.^2-2.\*x+2).^2;  >> q=integral(f,-inf,inf)  q = 1.5708 |  |  |  |  |  |  |  | | --- | --- | --- | --- | --- | --- | | **Sample Exercise Set on Improper Integral: 7.2**  Integration of the form  (i**mproper integral**)  1. Evaluate the following improper integral using Cauchy’s residue theorem (CRT):  (i) (ii)  (iii) (iv)  (v)  Laurent series generalize Taylor series. Indeed, whereas a Taylor series has positive integer powers  (and a constant term) and converges in a disc, a Laurent series is a series of positive and negative  integer powers of and converges in an annulus (a circular ring) with center . Hence by a  Laurent series we can represent a given function that is analytic in an annulus and may have  singularities outside the ring as well as in the “hole” of the annulus.  **Laurent’s Theorem:**  Let be analytic in a domain containing two concentric circles and with center , radii  and and the annulus between them. Then can be represented by the Laurent  series   |  |  | | --- | --- | | + |  |   The coefficients of Laurent series are given by the integrals  The variable of integration is denoted by , since is used in Laurent series.  The existing negative power of is known as **principal part**. If there is finite number of terms  in the principal part of in the Laurent series expansion then the coefficient of is called the  residue of at pole  **Laurent series expansion**  **Example:** **7.7** Obtain Laurent series expansion of  when , .  **Solution**: (i) Since  and  and  Let  At,  Equating coefficients of  Equating coefficients of              which is the required Laurent series.  (ii) For we have  Also            which is the required Laurent series.    **Sample Exercise Set on Laurent Series: 7.3**   1. State Laurent series. Expand in a Laurent series valid for   .   1. Expand in a Laurent series valid for   .   1. Expand in a Laurent series valid for      1. Find the function,  and the region of convergence for the following series:    6. Given functions (i) [Figure: (a) and (b)] and (ii)  .   [Figure: (a) and (c)] Determine the region of convergence and the series for the following figures:   |  |  |  | | --- | --- | --- | | (a)  C:\Users\Administrator\Desktop\C-1.jpg | (b)  C:\Users\Administrator\Desktop\c-2.png | (c) | | |